

HEAT EXCHANGE BETWEEN A WELL AND
FINELY-DISPERSED FROZEN GROUND

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The melting of finely-dispersed ground around a well or borehole is considered theoretically, and a method is proposed for the solution of this problem.

The intensive development of mineral sites in Siberia and the Far North has presented specialists with many new problems. One of the most important of these problems is that of the thermal interaction between boreholes or underground workings and the perpetually-frozen strata. Existing methods of calculating the heat transfer between wells or cylindrical workings and frozen ground [1-3] are based on the Stefan principle, in which it is assumed that all phase transitions take place on the zero isotherm, and the thermophysical properties of the phases are independent of temperature. These assumptions are valid for pure ice and coarsely-dispersed frozen ground, such as sand, but not for finely-dispersed frozen ground such as clays and loams. In these types of ground, some of the water (the free water) freezes (melts) sharply at 0°C, while the remaining water (the bound or combined water lying close to the walls of the capillaries) freezes (melts) over a wide range of negative temperatures [4]. One-dimensional linear problems of freezing and melting in finely-dispersed soils were solved in [5-7] with due allowance for this fact. In this paper we shall consider the axisymmetric problem of the melting of finely-dispersed frozen ground around a borehole or cylindrical working, and shall propose an approximate method of solving this problem.

Let us suppose that a flow of heat carrier at a temperature $T_0 > 0$ starts passing into a well of radius a (Fig. 1) surrounded by frozen ground with an initial temperature $T_M < 0$. As a result of this action, a melted region will arise close to the wall, its boundary gradually expanding with time. Considering that the phase transitions take place over the whole frozen zone and on the zero isotherm, we may write the heat-conduction equation for the frozen zone in the form

$$c_2 \rho_2 \frac{\partial}{\partial t} (\bar{T}_2) = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \bar{\lambda}_2 \frac{\partial \bar{T}_2}{\partial r} \right) + \rho_2 W_\Delta \frac{di}{dt}, \quad \bar{s} \leq \bar{r} < \infty,$$

$$\bar{T}_2|_{\bar{r}=\bar{s}} = 0, \quad \bar{T}_2|_{\bar{t}=0} = \bar{T}_M.$$

Remembering that the temperatures of frozen soils usually lie within the range 0 to -5°C, we may assume that the ice content and the ice-content-dependent thermophysical properties of the ground vary linearly with temperature:

$$i = i_0 + \bar{a} \bar{T}_2, \quad c_2 = c_2(0) + \bar{k} \bar{T}_2, \quad \bar{\lambda}_2 = \bar{\lambda}_2(0) + \bar{b} \bar{T}_2.$$

Let us transform to dimensionless variables

$$r = \frac{\bar{r}}{a}; \quad s = \frac{\bar{s}}{a}; \quad t = \frac{\bar{t}}{t_0}; \quad T_i = \frac{\bar{T}_i}{T_0}; \quad T_M = \frac{\bar{T}_M}{T_0};$$

$$i = 1, 2.$$

Thus the heat-conduction equation for the frozen zone takes the form

$$(1 + gT_2) \frac{\partial T_2}{\partial t} = \frac{\lambda_0}{r} \cdot \frac{\partial}{\partial r} \left(r(1 + bT_2) \frac{\partial T_2}{\partial r} \right), \quad s \leq r < \infty, \quad (1)$$

$$T_2|_{r=s} = 0; \quad T_2|_{t=0} = T_M; \quad (2)$$

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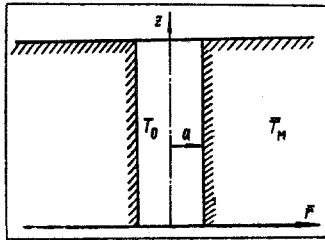


Fig. 1. Vertical section of the well; a is the radius of the well; T_0 is the temperature of the heat carrier inside the well; T_M is the initial temperature of the frozen ground; r, z are the cylindrical coordinates.

here

$$g = \frac{\bar{k}T_0}{c_2(0) - lW_{\Delta u}}; \quad \lambda_0 = \frac{\bar{\lambda}_2(0)t_0}{\rho_2 a^2 (c_2(0) - lW_{\Delta u})}; \quad b = \frac{\bar{b}T_0}{\bar{\lambda}_2(0)}.$$

The heat-conduction equation for such a region has the ordinary form:

$$\frac{\partial T_1}{\partial t} = \kappa_1 \left(\frac{1}{r} \cdot \frac{\partial T_1}{\partial r} + \frac{\partial^2 T_1}{\partial r^2} \right), \quad (3)$$

$$\frac{\partial T_1}{\partial r} \Big|_{r=a} = \alpha (T_1|_{r=a} - 1), \quad (4)$$

$$T_1|_{r=s} = 0; \quad (5)$$

here

$$\kappa_1 = \frac{\alpha_1 t_0}{a^2}; \quad \alpha = \frac{\bar{\alpha} a}{\bar{\lambda}_1}.$$

This system is completed by adding the Stefan condition on the zero isotherm, which in the present case reflects the absorption of the heat associated with the phase transition of the free water:

$$-\lambda_1 \frac{\partial T_1}{\partial r} \Big|_{r=s} + \lambda_2 \frac{\partial T_2}{\partial r} \Big|_{r=s} = \frac{ds}{dt}, \quad (6)$$

where

$$\lambda_1 = \frac{\bar{\lambda}_1(0) T_0 t_0}{\rho_2 l W_0 a^2}; \quad \lambda_2 = \frac{\bar{\lambda}_2(0) T_0 t_0}{\rho_2 l W_0 a^2}.$$

In order to solve the problem represented by Eqs. (1)-(6), we make use of the method of successive approximations [3, 8]. The application of this method to the problem of the melting of frozen ground around a well in the Stefan setting yields results agreeing closely with the exact solution of [9]. In view of the fact that the Stefan setting constitutes a limiting case of that represented by the set of equations (1)-(6) (with $g = b = 0$) we should expect that the results would agree closely with the exact solution in this case also.

We seek the solution to the problem (1)-(2) in the form

$$T_2 = T_2^{(0)} + T_2^{(1)}. \quad (7)$$

The function $T_2^{(0)}$ is defined as the solution of the steady-state problem

$$\frac{\partial}{\partial r} \left[r(1 + bT_2^{(0)}) \frac{\partial T_2^{(0)}}{\partial r} \right] = 0. \quad (8)$$

In view of the fact that the condition $\lim T_2 = T_M$ arising directly from the condition (2) cannot be satisfied by any solution of Eq. (8), we follow the usual procedure of introducing a radius of thermal influence $R(t)$, at which we have

$$T_2|_{r=R(t)} = T_M, \quad (9)$$

$$\frac{\partial T_2}{\partial r} \Big|_{r=R(t)} = 0. \quad (10)$$

Thus in order to construct an approximate solution we replace the boundary conditions (2) by the boundary conditions

$$T_2|_{r=s(t)} = 0, \quad T_2|_{r=R(t)} = T_M. \quad (11)$$

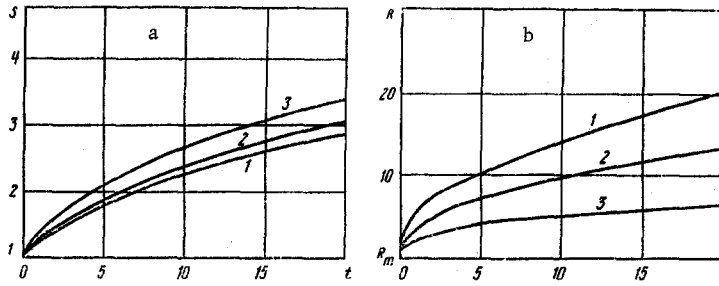


Fig. 2. Time dependence of the reduced radius of melting (a) and of the radius of thermal influence (b).

The function $T_2^{(0)}$ may be sought as a solution of the problem represented by Eqs. (8) and (11); it takes the form

$$T_2^{(0)} = \frac{1}{b} \sqrt{\frac{2bp}{\ln \frac{R}{s}} \ln \frac{r}{s} + 1} - \frac{1}{b} \cong p \frac{\ln \frac{r}{s}}{\ln \frac{R}{s}} - \frac{bp^2}{2} \cdot \frac{\ln^2 \frac{r}{s}}{\ln^2 \frac{R}{s}}, \quad (12)$$

where $p = T_M(1 + (b/2)T_M)$.

The latter transformation arises from the fact that the value of the product pb [4] is such as to make terms containing higher powers of this product negligible. The function $T_2^{(1)}$ is found by solving the problem

$$(1 + gT_2^{(0)}) \frac{\partial T_2^{(0)}}{\partial t} = \frac{\lambda_0}{r} \cdot \frac{\partial}{\partial r} \left(r(1 + bT_2^{(1)}) \frac{\partial T_2^{(1)}}{\partial r} \right), \quad (13)$$

$$T_2^{(1)}|_{r=s(t)} = 0, \quad T_2^{(1)}|_{r=R(t)} = 0. \quad (14)$$

On allowing for Eq. (12), we see clearly that the left-hand side of Eq. (13) is a known function of r , s , R , \dot{s} , and \dot{R} , and Eq. (13) may easily be integrated, giving the function $T_2^{(1)}(r, s, R, \dot{s}, \dot{R})$. Substituting the resultant expressions into (7) and satisfying condition (1), we obtain a relation of the form

$$\begin{aligned} \Phi_1(s, R, \dot{s}, \dot{R}) &= 0, \\ \dot{s} &\equiv \frac{ds}{dt}, \quad \dot{R} \equiv \frac{dR}{dt}, \end{aligned} \quad (15)$$

linear with respect to the derivatives. The second relation for these quantities may be derived from condition (6). For this we assume $T_2 \approx T_2^{(0)}$, $T_1 \approx T_1^{(0)}$, where $T_1^{(0)}$ is a solution to the problem:

$$\frac{1}{r} \cdot \frac{\partial T_1^{(0)}}{\partial r} + \frac{\partial^2 T_1^{(0)}}{\partial r^2} = 0, \quad (16)$$

$$T_1^{(0)}|_{r=s} = 0, \quad \frac{\partial T_1^{(0)}}{\partial r} \Big|_{r=1} = \alpha(T_1^{(0)}|_{r=1} - 1). \quad (17)$$

The solution is

$$T_1^{(0)} = \frac{\alpha \ln \frac{r}{s}}{1 + \alpha \ln s}. \quad (18)$$

On substituting these expressions into Eq. (6) we obtain a relation of the form

$$\Phi_2(s, R, \dot{s}) = 0. \quad (19)$$

Solving (15) and (19) with respect to the derivatives, we obtain a system of two ordinary differential equations of the first order

$$\dot{s} = \frac{\lambda_1 \alpha}{s(1 + \alpha \ln s)} + \frac{\lambda_2 p}{s \ln \frac{R}{s}}, \quad (20)$$

$$\dot{R} = -\lambda_0 R \left\{ \frac{p}{R^2(1 + bT_M)} + \frac{1}{\lambda_0 s} \left[\frac{\lambda_1 \alpha}{s(\alpha \ln s + 1)} + \frac{\lambda_2 p}{s \ln \frac{R}{s}} \right] \right\}$$

$$\times \sum_{i=1}^4 \left(\frac{I_{1i}}{\ln^{i-1} \frac{R}{s}} + \frac{(R^2 - s^2)}{4R^2} \cdot \frac{K_{1i}}{\ln^i \frac{R}{s}} \right) / \sum_{i=1}^4 \left(\frac{I_{2i}}{\ln^{i-1} \frac{R}{s}} + \frac{(R^2 - s^2)}{4R^2} \cdot \frac{K_{2i}}{\ln^i \frac{R}{s}} \right),$$

$$s|_{t=0} = 1; R|_{t=0} = R_m;$$

here

$$\begin{aligned} I_{11} &= 0, I_{21} = \frac{p^2(b-g) + gbp^3 - p}{2}, \\ I_{12} &= \frac{p^2(b-g) + gbp^3}{4} - \frac{p}{2}, I_{22} = \frac{p}{2} - \frac{3}{4} p^2(b-g) - gbp^3, \\ I_{13} &= -\frac{3}{4} (p^2(b-g) + gbp^3), I_{23} = \frac{3}{4} p^2(b-g) + \frac{3}{2} gbp^3, \\ I_{14} &= \frac{3}{2} gbp^3, I_{24} = -\frac{3}{2} gbp^3, \\ K_{11} &= p, K_{21} = 0, \\ K_{12} &= p^2(b-g) + p, K_{22} = -p, \\ K_{13} &= \frac{3}{4} (p^2(b-g) - gbp^3), K_{23} = -\frac{3}{2} p^2(b-g), \\ K_{14} &= -3gbp^3, K_{24} = 3gbp^3. \end{aligned}$$

R_m is the position of the radius of thermal influence at the instant at which the zero isotherm appears. The latter quantity is found from the equation $T_2^{(0)}|_{R=1} = 0$. From this we obtain

$$R_m = \exp(-T_{s1}/\alpha). \quad (21)$$

Solving the system (20) by means of a standard computer program for the integration of systems of ordinary differential equations, we obtain the law of motion of the zero isotherm $s(t)$ and the radius of thermal influence $R(t)$, and hence the temperature distributions in the melted and frozen zones at various instants of time.

Using the foregoing method, we calculated the heat-transfer process between a well and frozen clay soils with a total ice content of $W_0 + W_\Delta = 0.57$. We considered three cases:

- 1) $W_0 = 0.57, W_\Delta = 0;$
- 2) $W_0 = 0.47, \bar{u} = -0.05 \text{ deg}^{-1}, W_\Delta = 0.1;$
- 3) $W_0 = 0.27, \bar{u} = -0.15 \text{ deg}^{-1}, W_\Delta = 0.3.$

The values of the thermophysical constants were taken as follows:

$$\bar{\lambda}_1 = 1.279 \text{ W/m} \cdot \text{deg}, \bar{\lambda}_2(0) = 1.628 \text{ W/m} \cdot \text{deg}, C_2(0) = 2.13 \cdot 10^3 \text{ J/kg} \cdot \text{deg},$$

$$\bar{k} = 0.502 \cdot 10^3 \text{ J/kg} \cdot \text{deg}, \bar{b} = -0.204 \text{ J/kg} \cdot \text{deg}^2;$$

$$\alpha = 0.77, T_0 = 25 \text{ C},$$

$$t_0 = 20 \text{ h}, \rho_2 = 1500 \text{ kg/m}^3,$$

$$a = 0.14 \text{ m}, \bar{T}_M = -2 \text{ C},$$

$$l = 334.9 \cdot 10^3 \text{ J/kg}.$$

The first case corresponds to the Stefan setting (curve 1, Fig. 2); the second and third are distinguished by an increasing proportion of combined (bound) moisture (curves 2 and 3).

The results of our calculations of the velocity of the melting boundary and the radius of thermal influence are presented in Fig. 2 for all three cases.

On analyzing these curves, we may draw the following conclusions: the greater the proportion of combined moisture, the more rapidly does the melting boundary (zero isotherm) move, and the more

slowly does the boundary of thermal influence progress. Physically this may be explained in the following way. The velocity of the melting boundary is determined by the amount of ice transforming into water at this boundary. Whereas in the first case (Stefan setting) all the ice transforms into water on the zero isotherm, in the second and third cases some of the ice melts in the interval between the zero isotherm and the boundary of thermal influence. As a result of this the amount of ice melting on the zero isotherm diminishes and its rate of motion increases. For the same reason the velocity of the boundary of thermal influence diminishes.

NOTATION

a	is the radius of the well;
\bar{T}_M	is the initial temperature of the frozen ground;
T_0	is the temperature of the heat carrier in the well;
$\rho_i, c_i, \bar{\lambda}_i, a_i$	are the density, specific heat, thermal conductivity, and thermal diffusivity of the ground;
$\bar{\alpha}$	is the heat-transfer coefficient;
W_0, W_Δ	are the relative quantities of free and combined (bound) moisture;
i	is the relative ice content;
l	is the heat of the phase transition;
t_0	is the time scale;
t	is the time;
r	is the cylindrical coordinate;
\bar{T}_i	is the temperature of the ground;
s	is the radius of melting.

Subscripts

$i = 1, 2$ denote the melted and frozen ground, respectively.

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